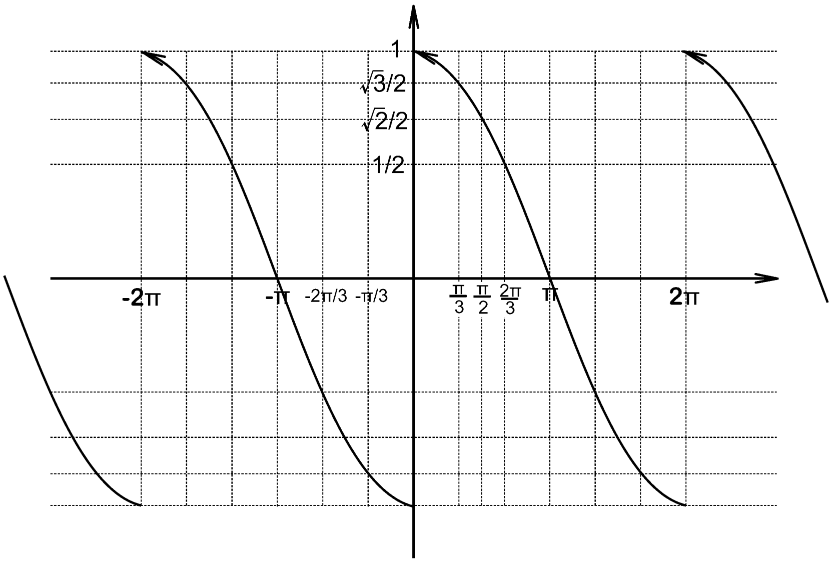
Pismeni ispit iz Analize III, 10.06.2013.
ispit pisati isključivo hemiskom olovkom



1. Funkciju definisanu grafikom razviti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda

$$\frac{1}{1\cdot 3}-\frac{3}{5\cdot 7}+\ldots+ \frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+\ldots.$$

2. (25%) (a) Odrediti ekstreme funkcije $f(x,y)=x^2-xy+y^2-2x-2y$.
(75%) (b) Izračunati integral

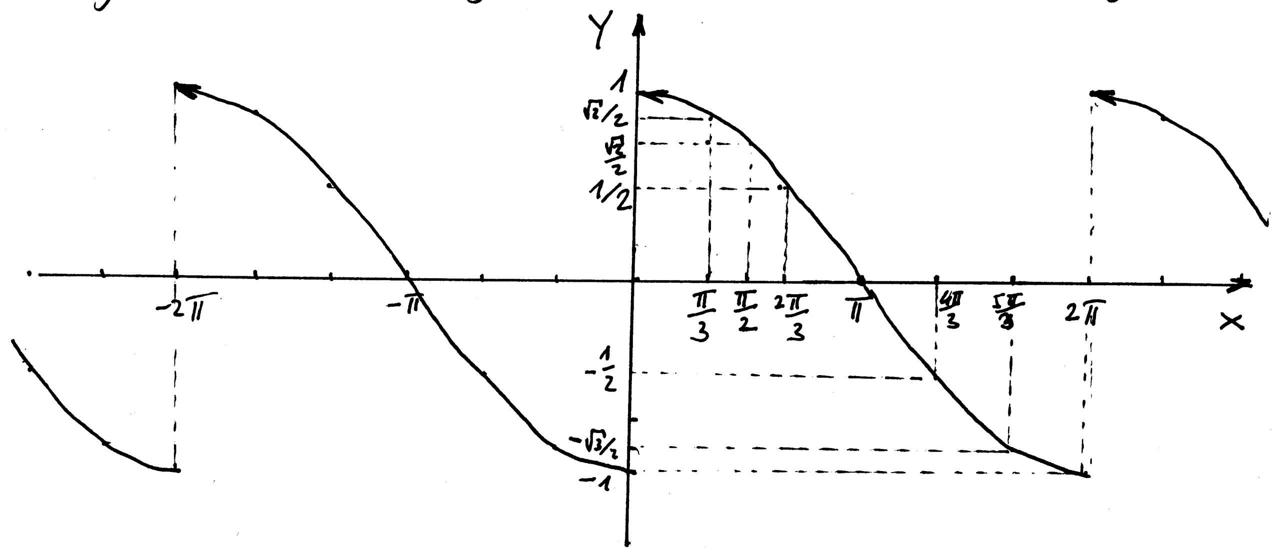
$$\iiint_\Omega \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

gdje je $\Omega=\{(x,y,z)\in\mathbb{R}^3 \,|\, x^2+y^2+z^2\le z,\ x^2+y^2\le z^2\}$.

3. Izračunati krivoliniski integral druge vrste $I=\oint_C (y-z)\mathrm{d}x+(z-x)\mathrm{d}y+(x-y)\mathrm{d}z$ gdje je C krug $x^2+y^2+z^2=a^2$ $(a>0)$, $y=x\,\mathrm{tg}\,\alpha$, $(0<\alpha<\frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljnke na satu ako se posmatra sa pozitivnog dijela x-ose.

4. Izračunati površinski integral $I=\iint_S \frac{\mathrm{d}S}{(1+z)^2}$ ako je S sfera $x^2+y^2+z^2=1$.

Funkciju definisanu grafikom razviti u Furijerov red



Dobijeni rezultat iskoristiti za sumiranje reda

$$\frac{1}{1\cdot 3}-\frac{3}{5\cdot 7}+\dots+\frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+\dots$$

Rj. Prikazana f-ja je periodična, perioda 2π pa je možemo pretvoriti u Furijer-ov red. Datu f-ju označimo sa $y=f(x)$. Primjetimo sa slike da imamo $f\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$, $f\left(\frac{\pi}{2}\right)=\frac{\sqrt{2}}{2}$ $f\left(\frac{2\pi}{3}\right)=\frac{1}{2}$, $f(\pi)=0$.
Kako je $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$, $\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$, $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$, $\cos\left(\frac{\pi}{2}\right)=0$
to možemo primjetiti da je data f-ja $y=\cos\left(\frac{x}{2}\right)$, $0\le x\le 2\pi$
Trigonometriski red oblika $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$
$x\in[a,b]$ nazivamo Furijer-ov red na intervalu $[a,b]$ gdje su

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx,$$

$n=1,2,\dots$ Furijer-ovi koeficijenti. U našem slučaju posmatramo interval $[0,2\pi]$ pa imamo $b-a=2\pi$, $\frac{2}{b-a}=\frac{1}{\pi}$, $\frac{2n\pi x}{b-a}=nx$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}\cos\frac{x}{2}\,dx=\frac{1}{\pi}2\int_0^{2\pi}\cos\frac{x}{2}\,d\left(\frac{x}{2}\right)=\frac{2}{\pi}\sin\frac{x}{2}\Big|_0^{2\pi}=0$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\left|\begin{matrix}\text{data}\\ \text{f-ja } f(x)\\ \text{je periodična}\end{matrix}\right|=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\left|\begin{matrix}\text{prema datoj slici}\\ f(x)\text{ je neparna.}\\ \text{Kako je }\cos nx\\ \text{parna to je}\\ f(x)\cos nx\text{ neparna}\\ \text{f-ja, i imamo}\\ \text{simetričan interval}\end{matrix}\right|$$

$$=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\cos\left(\frac{x}{2}\right)\sin nx\,dx=\left|\begin{matrix}\sin(A+B)=\sin(A)\cos B+\sin B\cos A\\ +\ \sin(A-B)=\sin A\cos B-\sin B\cos A\\ \hline 2\sin A\cos B=\sin(A+B)+\sin(A-B)\end{matrix}\right|$$

$$=\frac{1}{\pi}\cdot\frac{1}{2}\int_0^{2\pi}\left(\sin\left(nx+\frac{x}{2}\right)+\sin\left(nx-\frac{x}{2}\right)\right)dx=\frac{1}{2\pi}\int_0^{2\pi}\left(\sin\left(n+\frac{1}{2}\right)x+\sin\left(n-\frac{1}{2}\right)x\right)dx$$

$$=\frac{1}{2\pi}\cdot\frac{(-1)}{n+\frac{1}{2}}\cos\left(n+\frac{1}{2}\right)x\Big|_0^{2\pi}+\frac{1}{2\pi}\cdot\frac{(-1)}{n-\frac{1}{2}}\cos\left(n-\frac{1}{2}\right)x\Big|_0^{2\pi}=$$

$$=\frac{(-1)}{\pi(2n+1)}\left(\overbrace{\cos(2n+1)\pi-\cos 0}^{-2}\right)+\frac{(-1)}{\pi(2n-1)}\left(\overbrace{\cos(2n-1)\pi-\cos 0}^{-2}\right)=\frac{2}{\pi(2n+1)}+\frac{2}{\pi(2n-1)}$$

$$=\frac{8n}{\pi(2n-1)(2n+1)}$$

Prema tome $\cos\frac{x}{2}\sim\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}\sin nx$ traženi Furijerov red

Ako za x uzmemo $\frac{\pi}{2}$ imamo

$$\frac{8}{\pi}\left(\frac{1}{1\cdot 3}+\frac{2\cdot 0}{3\cdot 5}+\frac{3\cdot(-1)}{5\cdot 7}+\ldots+\frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+\ldots\right)=\cos\frac{\pi}{4}\quad\frac{\sqrt{2}}{2}$$

Prema tome

$$\frac{1}{1\cdot 3}-\frac{3}{5\cdot 7}+\ldots+\frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+\ldots=\frac{\pi\sqrt{2}}{16}$$ tražena suma

Odrediti ekstreme f-je

$$f(x,y) = x^2 - xy + y^2 - 2x - 2y.$$

Rj. Odredimo parcijalne izvode

$$\frac{\partial f}{\partial x} = 2x - y - 2$$

$$\frac{\partial f}{\partial y} = -x + 2y - 2$$

Zatim da bi odredili stacionarne tačke riješimo sistem

$$2x - y - 2 = 0$$
$$-x + 2y - 2 = 0 \quad / \cdot 2$$

$$2x - y - 2 = 0$$
$$+ \quad -2x + 4y - 4 = 0$$

$$3y - 6 = 0$$
$$y = 2$$

$$y = 2 \Rightarrow 2x - 2 - 2 = 0$$
$$x = 2$$

Stacionarna tačka je $M(2,2)$.

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

Za $M(2,2)$ imamo

$A = 2, \; B = -1, \; C = 2$

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = 4 - 1 = 3 > 0$$

f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$$f_{min}(2,2) = 4 - 4 + 4 - 4 - 4 = -4$$

(#) Izračunati integral

$$\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz$$

gdje je $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq z,\ x^2+y^2 \leq z^2\}$.

Rj.

$x^2+y^2+z^2 = z$ je jednačina sfere

$x^2+y^2 = z^2$ je jednačina čunja

Odmah vidimo da čunj ima vrh u koordinatnom početku.

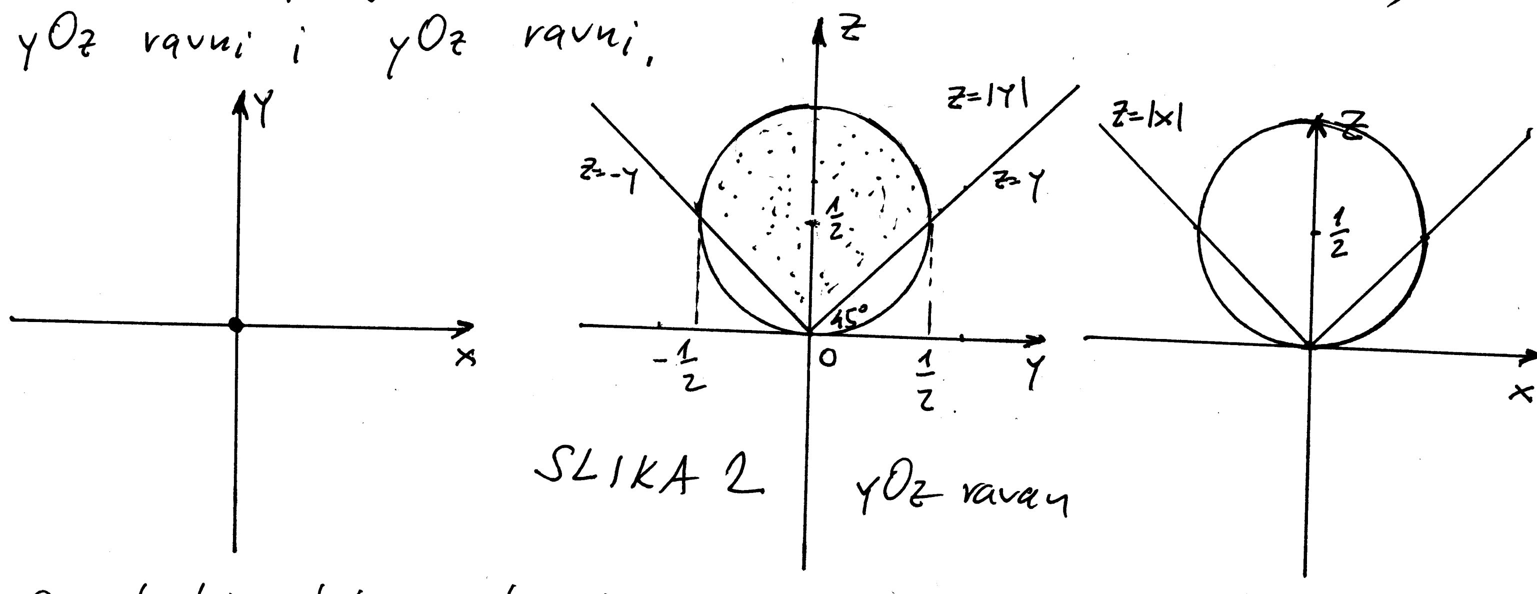
$$x^2+y^2+z^2 = z$$

$$x^2+y^2+z^2 - 2\cdot z\cdot \frac{1}{2} + \frac{1}{4} = \frac{1}{4}$$

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

centar sfere je u tački $(0,0,\frac{1}{2})$ a poluprečnik $\frac{1}{2}$

Napravimo presjeke datih figura redom sa xOy ravni, yOz ravni i yOz ravni.



SLIKA 2 yOz ravan

Sa datih slika odmah vidimo da je integral skroz teško izračunati uz pomoć pravougaonih koordinata. Uvedimo sferne koordinate.

$x = r\sin\varphi\cos\theta$

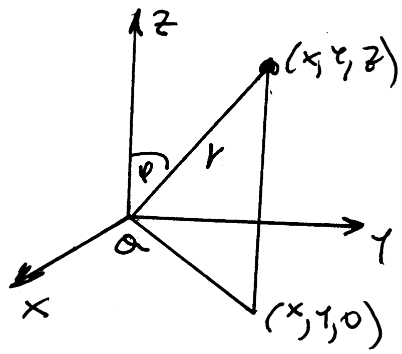
$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\, dr\,d\varphi\,d\theta$

$\Omega \xrightarrow{\text{transformiramo}} \Omega'$

opis točke



Sa slike 2 čitamo granice za φ i θ. Granice za r možemo odrediti na osnovu formule

$$\Omega' : \begin{cases} 0 \le r \le \cos\varphi \\ 0 \le \varphi \le \frac{\pi}{4} \\ 0 \le \theta \le 2\pi \end{cases}$$

$x^2+y^2+z^2 \le z$

tj. $r^2 \le r\cos\varphi \quad /:r$

$r \le \cos\varphi$.

$$\boxed{x^2+y^2+z^2 = \ldots = r^2}$$

$$\iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \left| \begin{array}{l} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{array} \right| = \iiint\limits_{\Omega'} r^3 \sin\varphi\, dr\,d\varphi\,d\theta$$

$$= \int_0^{\pi/4} \sin\varphi\,d\varphi \int_0^{\cos\varphi} r^3\,dr \int_0^{2\pi} d\theta = 2\pi \int_0^{\pi/4} \sin\varphi\,d\varphi \int_0^{\cos\varphi} r^3\,dr = 2\pi\cdot\frac{1}{4}\int_0^{\pi/4} \sin\varphi\cos^4\varphi\,d\varphi$$

$$= \left| \begin{array}{l} d(\cos\varphi) = -\sin\varphi\,d\varphi \\ \sin\varphi\,d\varphi = -\,d(\cos\varphi) \end{array} \right| = -\frac{\pi}{2}\int_0^{\pi/4} \cos^4\varphi\, d(\cos\varphi) = -\frac{\pi}{2}\cdot\frac{1}{5}\cos^5\varphi \Big|_0^{\frac{\pi}{4}} =$$

$$= -\frac{\pi}{10}\left(\left(\frac{\sqrt{2}}{2}\right)^5 - 1\right) = \frac{\pi}{10}\left(1 - \frac{\sqrt{2}}{8}\right)$$ traženo rješenje

(#) Izračunati krivolinijski integral druge vrste

$$I = \oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$$ gdje je C krug

$x^2+y^2+z^2=a^2$ $(a>0)$, $y = x\,tg\alpha$, $(0<\alpha<\frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljke na satu ako se posmatra sa pozitivnog dijela x-ose.

Rj.

$$C: \begin{cases} x^2+y^2+z^2=a^2, (a>0) \\ y = x\,tg\alpha \quad (0<\alpha<\frac{\pi}{2}) \end{cases}$$

Parametrizujmo krivu C. Kako je $tg\alpha = \frac{\sin\alpha}{\cos\alpha}$ to možemo npr. uzeti $x = a\cos\alpha\sin\varphi$. Tada,

$$y = x\,tg\alpha = a\cancel{\cos\alpha}\sin\varphi\frac{\sin\alpha}{\cancel{\cos\alpha}} = a\sin\alpha\sin\varphi$$

Dalje iz $x^2+y^2+z^2=a^2$ imamo

$$(a\cos\alpha\sin\varphi)^2+(a\sin\alpha\sin\varphi)^2+z^2=a^2$$

$$a^2\underbrace{(\cos^2\alpha+\sin^2\alpha)}_{=1}\sin^2\varphi+z^2=a^2$$

$$z^2 = a^2 - a^2\sin^2\varphi$$

$$z^2 = a^2(1-\sin^2\varphi) \Rightarrow z = a\cos\varphi$$

Dati krug C ima sljedeću parametrizaciju

$$C: \begin{cases} x = a\cos\alpha\sin\varphi \\ y = a\sin\alpha\sin\varphi \\ z = a\cos\varphi \\ 0 \le \varphi \le 2\pi \end{cases} \qquad \begin{aligned} dx &= a\cos\alpha\cos\varphi\,d\varphi \\ dy &= a\sin\alpha\cos\varphi\,d\varphi \\ dz &= -a\sin\varphi \end{aligned}$$

$$\oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz =$$

$$= \int_0^{2\pi} \Big[(a\sin\alpha\sin\varphi - a\cos\varphi)\, a\cos\alpha\cos\varphi + (a\cos\varphi - a\cos\alpha\sin\varphi)\, a\sin\alpha\cos\varphi$$

$$+ (a\cos\alpha\sin\varphi - a\sin\alpha\sin\varphi)(-a)\sin\varphi \Big]\, d\varphi$$

$$= a^2\sin\alpha\cos\alpha \int_0^{2\pi} \sin\varphi\cos\varphi\, d\varphi - a^2\cos\alpha \int_0^{2\pi} \cos^2\varphi\, d\varphi + a^2\sin\alpha \int_0^{2\pi} \cos^2\varphi\, d\varphi$$

$$- a^2\sin\alpha\cos\alpha \int_0^{2\pi} \sin\varphi\cos\varphi\, d\varphi - a^2\cos\alpha \int_0^{2\pi} \sin^2\varphi\, d\varphi + a^2\sin\alpha \int_0^{2\pi} \sin^2\varphi\, d\varphi$$

$$= -a^2\cos\alpha \int_0^{2\pi} \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1}\, d\varphi + a^2\sin\alpha \int_0^{2\pi} \underbrace{(\sin^2\varphi + \cos^2\varphi)}_{=1}\, d\varphi$$

$$= 2\pi a^2 (\sin\alpha - \cos\alpha) = 2a^2(\sin\alpha - \cos\alpha)\pi$$

traženo rješenje

(#) Izračunati površinski integral

$$I=\iint_S \frac{dS}{(1+z)^2}$$

ako je S sfera $x^2+y^2+z^2=1$.

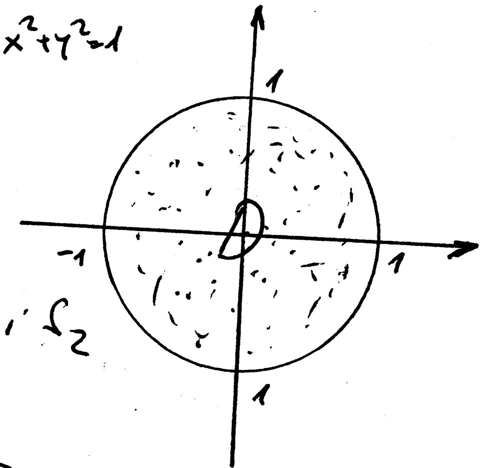
Rj. Zadatak se može uraditi na više načina

I način

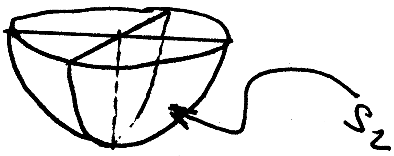
$z^2=1-x^2-y^2$

$z=\pm\sqrt{1-x^2-y^2}$

Presjek sfere $x^2+y^2+z^2=1$ sa xOy-ravni je krug $x^2+y^2=1$



U ovom slučaju sferu S ćemo podjeliti na dvije polusfere S_1 i S_2



$$I=\iint_S \frac{dS}{(1+z)^2}=\iint_{S_1}\frac{dS_1}{(1+z)^2}+\iint_{S_2}\frac{dS_2}{(1+z)^2}$$

gdje je

$S_1: z=\sqrt{1-x^2-y^2}$

a $S_2 = -\sqrt{1-x^2-y^2}$

Znamo da je $dS_1=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$

$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{1-x^2-y^2}}$, $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{1-x^2-y^2}}$, $dS_1=\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\,dxdy$

$$dS_1=\sqrt{\frac{1}{1-x^2-y^2}}=\frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$$

Sad imamo

$$\iint_{S_1} \frac{dS_1}{(1+z)^2} = \iint_D \frac{1}{\left(1+\sqrt{1-x^2-y^2}\right)^2} \cdot \frac{1}{\sqrt{1-x^2-y^2}}\, dx\, dy =$$

$$= \left| \begin{array}{l} \text{uvodimo polarne koordinate} \\ x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ dx\,dy=\rho\, d\rho\, d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D_1 : \begin{cases} 0\le\rho\le 1 \\ 0\le\varphi\le 2\pi \end{cases} \quad x^2+y^2=\ldots=\rho^2 \right| = \iint_{D_1} \frac{\rho\, d\rho\, d\varphi}{\left(1+\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}}$$

$$= \int_0^1 \frac{\rho\, d\rho}{\left(1+\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}} \int_0^{2\pi} d\varphi = \left| \begin{array}{l} 1-\rho^2=t^2 \\ -2\rho\, d\rho = 2t\, dt \\ \rho\, d\rho = -t\, dt \end{array} \right| = 2\pi \int_1^0 \frac{-t\, dt}{(1+t)^2\cdot t}\, dt = \ldots = \pi$$

Slično

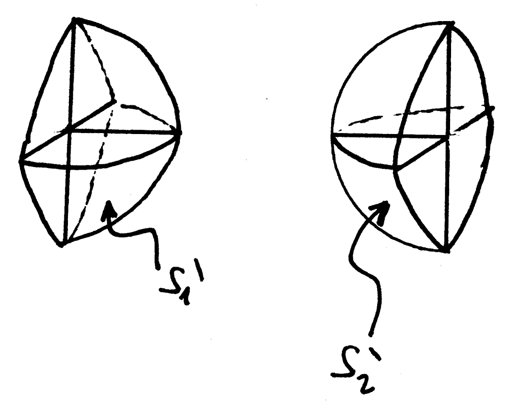
$$\iint_{S_2} \frac{dS_2}{(1+z)^2} = \iint_D \frac{1}{\left(1-\sqrt{1-x^2-y^2}\right)} \cdot \frac{dx\, dy}{\sqrt{1-x^2-y^2}} = \left| \begin{array}{l} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{array} \right| = \iint_{D_1} \frac{\rho\, d\rho\, d\varphi}{\left(1-\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}}$$

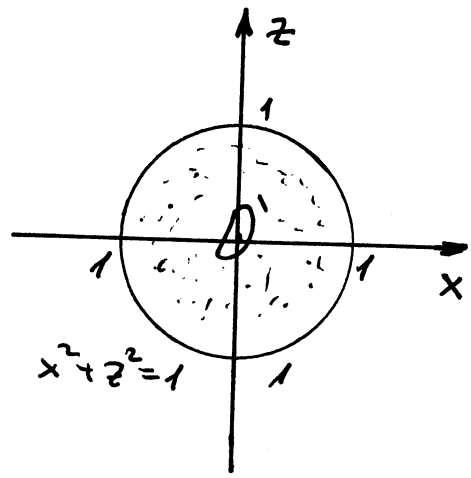
$$= 2\pi \int_0^1 \frac{\rho\, d\rho}{\left(1-\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}} = \left| \begin{array}{l} 1-\rho^2=t^2 \\ -2\rho\, d\rho=2t\, dt \\ \rho\, d\rho=-t\, dt \end{array} \right| = 2\pi \int_0^1 \frac{t\, dt}{(1-t)^2\, t} = \ldots = \infty$$

II način

$$y^2 = 1-x^2-z^2$$

$$y = \pm\sqrt{1-x^2-z^2}$$





$$I = \iint_S \frac{dS}{(1+z)^2} = \iint_{S_1'} \frac{dS_1'}{(1+z)^2} + \iint_{S_2'} \frac{dS_2'}{(1+z)^2}$$

gdje je $S_1' : y=\sqrt{1-x^2-z^2}$

$S_2' : y=-\sqrt{1-x^2-z^2}$

$$y=\sqrt{1-x^2-z^2}, \quad \frac{\partial y}{\partial x} = \frac{-x}{\sqrt{1-x^2-z^2}}, \quad \frac{\partial y}{\partial z} = \frac{-z}{\sqrt{1-x^2-z^2}}, \quad 1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2 = \frac{1}{1-x^2-z^2}$$

$$\iint_{S_1'} \frac{dS_1'}{(1+z)^2} = \iint_{D} \frac{1}{(1+z)^2} \cdot \frac{dx\, dz}{\sqrt{1-x^2-z^2}} = \ldots$$

Slično i za S_2'